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COORDINATING THE MOTION OF SEVERAL DISCS

by

Chee K. Yap

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Coordinating the Motion of Several Discs

Chee K. Yap

Courant Institute of Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012

ABSTRACT

We give an $O(n^2)$ and an $O(n^3)$ time algorithm for the problem of coordinating the motion of two and three discs (respectively) moving amidst polygonal obstacles. Our approach is based on the idea of retraction, and improves previous algorithms by Schwartz and Sharir.

Key Words: robotics, motion planning, coordinated motion, efficient algorithms, retraction approach.

1. Introduction

Various computational problems arise in robotics in the general area of motion planning [Bra]. Although motion planning problems have been studied for many years from the viewpoint of heuristic programming or artificial intelligence (eg. [N]) and mechanics (eg. [P]), it is only recently that algorithmic and complexity theoretic techniques have been brought to bear. For recent progress along this line, see [SS1,SS2,HJW,OY,OSY]. The *many body problem* in motion planning considered in this paper is posed as follows:

> Given m bodies $\mathbf{B} = (B_1, B_2, \ldots, B_m)$, a set of obstacles S, an initial placement $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and a final placement $\mathbf{q} = (q_1, \ldots, q_m)$ of $\mathbf{B}$, the problem is to plan a (coordinated) motion of $\mathbf{B}$ from $\mathbf{p}$ to $\mathbf{q}$ (if one exists).

A variety of problems arise depending on whether one considers this question in two or three dimensions, whether each B_i is rigid or jointed, whether $\mathbf{B}$ is fixed or varies with the input, and whether the obstacles S are assumed to be polyhedral or have more general shapes. The results of [SS2] imply that for any fixed $\mathbf{B}$, where the bodies and obstacles are bounded by algebraic surfaces with bounded degrees, the problem is polynomial time solvable. However, for each specific $\mathbf{B}$, it remains of interest to obtain algorithms more efficient than that which follows from the general results in [SS2].

The case where $m = 1$ has been extensively studied (see the papers cited above). The case $m > 1$ is relatively new and [SS3] appears to be the first purely algorithmic study of this sort. Note that the only mutual constraint among the bodies in $\mathbf{B}$ in a 'coordinated motion' is non-collision among them (compare this with the mutual constraints among the different links of a jointed robot arm). An instance of coordination is the important but little-understood problem of making two robot arms cooperate on a task. Another example is that of coordinating many roving robots in a controlled environment such as an automated warehouse or in a factory.

We shall consider the simplest case where the bodies are discs moving amidst polygonal obstacles in the plane. Note that the planar version of the problem is quite realistic since in practice, a roving robot would represent its world by a floor-plan with the robot itself being approximated by an enclosing disc. For example, the robot vehicle in [M] employs a disc algorithm for moving amidst circular obstacles running in $O(n^4)$ time. (However, we remark this can easily be improved to $O(n\log^2 n)$ by adapting the algorithm in [OY] and a result of M. Sharir that the Voronoi diagram of a collection of circles can be computed in $O(n\log^2 n)$ time.) Another reason for considering discs rather than more complex shapes is that they give rise to comparatively simpler algorithms (cf. [OSY1,OSY2]), and thus disc algorithms should probably be used as an initial heuristic in finding a path. In [SS3], an $O(n^3)$ (resp. $O(n^{13})$) time algorithm for coordinating the motion of two (resp. three) discs were found. In this paper, we shall present simpler algorithms for the same problems with running times of $O(n^2)$ and $O(n^3)$, respectively. Let us remark that in general, it is not obvious that coordinating k discs should take $O(n^k)$ time (since it seems that most problems with $2k$ degrees of freedom -- as is the case with k discs -- takes $\Omega(n^{2k})$ time). Figures 1 and 2 illustrate two problems involving the coordination of several discs.

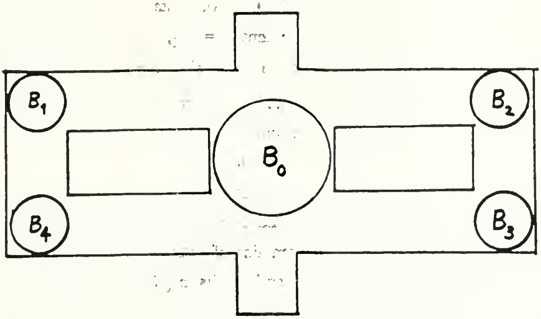


Figure 1. Exchange the placements of B_1 and B_3, keeping the other discs (eventually) in their original placements: thus B_0 acts as a 'sliding door'.

The algorithms we present are based on the idea of 'retraction': intuitively, we reduce (via a suitable map call the 'retraction' [†]) the search of a motion in the space FP of *free placements* (ie. placements where the body is not incident on any obstacle) to the search of another 'motion' in a lower dimensional subspace (the 'retract') of FP. This approach, first described in [OY,OSY1-3], has led to very efficient algorithms. We should note that in these previous papers, the 'retract' used is a suitable generalization of *Voronoi diagrams* [K]. The use of Voronoi diagrams in motion-planning seems to be first proposed by [R] who approached it as a problem of artificial intelligence and treats a digitalized space. One contribution of the present paper is to show that the 'retraction'

[†] In topology, a 'retraction' is a continuous map. We relax the continuity requirement: the important property for us is that the map preserves the connectivity structure of FP in a sense to be made precise.

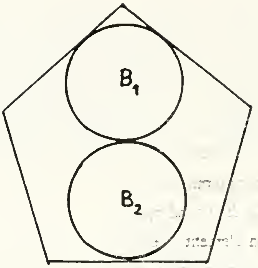


Figure 2. Exchange the placements of B_1 and B_2: tight cooperation between the two discs is necessary.

approach can be applied without resorting to (generalizations of) Voronoi diagrams.

On the inherent complexity of the Many Body Problem: All the polynomial time algorithms cited above for coordinating motion assume that **B** is fixed, i.e. the number of bodies and their shapes are fixed. If **B** is allowed to vary with the input, the problem of moving many discs amidst polygonal obstacles is *NP*-hard [SY]. If discs are replaced by rectangular bodies, [HSS] shows that the problem is *PSPACE*-hard. Observe that rectangles are apparently harder than discs since they are 2-parameter objects and they move in a 3-dimensional space. (Discs have one parameter and move in a 2-dimensional space.)

This paper is organized as follows: after some preliminary definitions in section 2, we introduce the retraction map for the problem of coordinating two discs in section 3. This retraction reduces a coordinated motion to a so-called 'stable motion'. A stable motion can be decomposed into a sequence of 'elementary' motions together with certain discontinuities. Section 4 analyzes the elementary motions and section 5 the discontinuities. This leads to a combinatorial graph *CCG* whose edges correspond to the elementary motions and canonical forms of the discontinuities. Finding a coordinated motion is thus reduced to finding a path in *CCG*; the algorithm for constructing and searching *CCG* is described in section 6. In section 7 we outline the solution for three discs. Section 8 concludes.

2. Preliminary

Let S, the *obstacle set*, be a closed subset of the plane bounded by polygonal lines. (Note that S may be disconnected.) Let Ω be the complement of S and for simplicity, we assume Ω is bounded (our results can be modified to hold even if Ω is unbounded). Let B $= (B_1, \ldots, B_m)$ where B_i is an (open) disc of radius r_i. A *placement* p $= (p_1, \ldots, p_m)$ of B is a sequence of m points in Ω where p_i specifies the center of B_i. B is assumed fixed (with $m = 2$ or 3 in this paper) so that the input to the problem is a description of S and the initial and final placements of B. As in [K], the boundary of S is decomposed into a collection of pairwise disjoint set of *elements* where an element is either an open line segment (called a 'wall') or a point ('corner'). The *input size* is proportional to n, the number of corners in the boundary of S. For any point x, let $B_i(x)$ denote the set of points y occupied by B_i when placed at x, ie., the distance between y and x is $< r_i$. The placement p is *free* (for B) if the $m+1$ sets

$$S,\ B_1(p_1), \cdots, B_m(p_m)$$

are pairwise disjoint. Let $FP = FP(\mathrm{B})$ denote the set of free placements of B. Similarly, $FP(B_i)$ denotes the set of free placements x of B_i, ie., where S and $B_i(x)$ are disjoint.

Note that $FP(B_i)$ is a closed planar set. It is easy to see that the boundary of $FP(B_i)$ can decomposed into a collection of (closed) line segments and (closed) circular arcs. Following [SS3], we call the line segments *displaced walls* and the arcs *displaced corners*. See Figure 3.

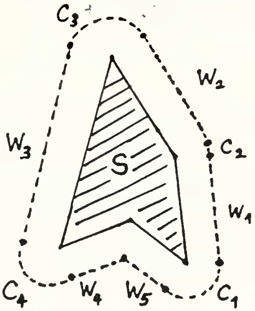


Figure 3. Four displaced corners and five displaced walls.

Clearly the displaced elements (of a fixed B_i) may intersect at their endpoints. To simplify the paper, we make two assumptions: (a) no interior point of a displaced element is shared by another displaced element, and (b) no point is common to three displaced elements. These assumptions are equivalent to saying that each connected component of the boundary of $FP(B_i)$ is a simple closed path. Let G_i be the planar graph representing the boundary of $FP(B_i)$: the *edges* of G_i are the displaced elements and its *vertices* are endpoints of edges. Clearly G_i has $O(n)$ edges and vertices. Recall the notion of a Voronoi diagram $Vor(S)$ of S [OSY,Ki]. [Ki] shows that $Vor(S)$ can be computed in $O(n\log n)$ time. The following lemma implies that G_i can be computed in $O(n\log n)$ time also.

Lemma 1. Given $Vor(S)$, the graph G_i can be computed in linear time.

A sketch proof goes as follows. For each Voronoi edge e, it is easy to determine in constant time the intersection points of e with the boundary of $FP(B_i)$: in fact since each Voronoi edge is either straight or parabolic, there can be at most two intersection points. Let V_i be the set of these intersection points. It is not hard to see that V_i is precisely the vertex set of G_i. It remains to determine the edges of G_i. Here we use the fact that two vertices u and v are adjacent in G_i iff they lie on the boundary of a common Voronoi cell. We can process the Voronoi edges in such a way that each edge is examined a constant number of times, resulting in a linear time algorithm.

By a *motion* of B_i we mean a function π_i: $[0,1] \rightarrow FP(B_i)$. Note that (it is technically convenient that) π is not required to be continuous although, ultimately, only continuous motions are of interest to us. By a *coordinated motion* of B we mean $\pi = (\pi_1, \ldots, \pi_m)$ where each π_i is a motion of B_i and $\pi(t)$ $(= (\pi_1(t), \cdots, \pi_m(t)))$ is free for all t in $[0,1]$. We abbreviate 'coordinated motion' to 'co-motion'.

3. Coordination of Two Discs

Let us first informally describe the retraction for two discs. Given any initial placement $\mathbf{p} = (p_1, p_2)$ of $\mathbf{B} = (B_1, B_2)$, we imagine a mutually repulsive force between the discs which pushes them as far apart as possible (ie. until each is backed-up against the obstacle S): such a 'backed-up' placement $Im(\mathbf{p})$ will be called 'stable'. Note that since Ω is bounded, $Im(\mathbf{p})$ is well-defined for all $\mathbf{p}$. This transformation can be applied pointwise to a coordinated motion π of B resulting in a 'stable' co-motion $Im(\pi)$. One difficulty that must be faced is that $Im(\pi)$ may not be continuous even though π is continuous. We begin with the definition of the retraction map.

Definition. For $\mathbf{p} = (p_1, p_2) \in FP$, its *image* $Im(\mathbf{p})$ is defined to be $\mathbf{q} = (q_1, q_2)$ where q_i $(i = 1, 2)$ is the first point on the half-line from p_i directed away from p_{3-i} such that q_i lies on the boundary of $FP(B_i)$ and either the line $p_{3-i}q_i$ is tangential at q_i to some displaced corner (not wall) e of G_i or for all $\delta > 0$ small enough,

$$q_i + \delta\cdot(q_i - p_{3-i}) \notin FP(B_i). \tag{3}$$

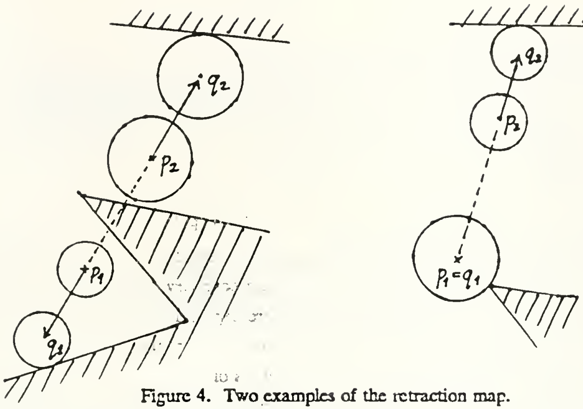


Figure 4. Two examples of the retraction map.

Note that q_i may be an endpoint of e in the above definition. The condition (3) says that if we try to 'push' B_i at q_i any further away from B_{3-i}, we would run into the obstacle. Note that our definition of *Im* implies that the 'repulsive force' is not obstructed by obstacles that intervene between p_1 and p_2. For any free placement $\mathbf{p} = (p_1, p_2)$, we refer to the line determined by p_1 and p_2 as the *line through* $\mathbf{p}$.

> **Definition.** A placement $\mathbf{p}$ is *stable* if there exists some free placement $\mathbf{q}$ such that $\mathbf{p} = Im(\mathbf{q})$. A co-motion π is *stable* if there exists a continuous co-motion π' such that $\pi = Im(\pi')$. A stable placement $\mathbf{p} = (p_1, p_2)$ is *critical* if more than one of the conditions (1)-(4) below holds:
>
> (1). p_1 is a vertex (of G_1).
>
> (2). p_2 is a vertex (of G_2).
>
> (3). The line through $\mathbf{p}$ is tangential at p_1 to some displaced corner (not wall) e of G_1.
>
> (4). The line through $\mathbf{p}$ is tangential at p_2 to some displaced corner (not wall) e of G_2.

Motion in *FP*, which has four degrees of freedom, is reduced by the retraction *Im* to stable motion with two degrees of freedom. Our analysis in the next section can be seen to involve further retractions of the stable placements to the critical placements, ie. a reduction of a space with two degrees of freedom to another space with zero degrees of freedom.

Lemma 2. There are $O(n^2)$ critical placements.

Proof. There are clearly $O(n^2)$ critical placements which are formed by pairs of vertices. To count the other type of critical placements, we note that each displaced corner e may be associated with $O(n)$ critical placements $\mathbf{p}$ such that the line through $\mathbf{p}$ is tangential to e. **Q.E.D.**

4. Elementary Coordinated Motion

To analyze stable co-motions, first consider the simplest sort of stable co-motion. If X_i are sets of points, it is convenient to say that a placement (p_1, p_2) *belongs to* (X_1, X_2) if $p_i \in X_i$. Typically, X_i is a displaced element. Let $\mathbf{e} = (e_1, e_2)$ where e_i is a displaced element (ie. an edge) of G_i.

> **Definition.** A stable co-motion π is said to be *elementary* if it is continuous and there exists a pair $\mathbf{e}$ of displaced elements such that $\pi(t)$ belongs to $\mathbf{e}$ for all t in [0,1]. Two stable free placements $\mathbf{p}$ and $\mathbf{q}$ are *adjacent* if there is an elementary co-motion between them.
>
> **Lemma 3.** Let $\mathbf{e} = (e_1, e_2)$ be a pair of displaced elements. Every stable placement $\mathbf{p} = (p_1, p_2)$ belonging to $\mathbf{e}$ is adjacent to some critical placement belonging to $\mathbf{e}$.

Proof. The proof is by a fairly straighforward construction as follows: keeping p_1 fixed, move p_2 along e_2 in a direction such that the separation between the discs is increasing until the first time that p_2 reaches a vertex or the line through $\mathbf{p}$ is a tangent to e_i (for some $i = 1$ or 2). If the former, we now keep p_2 fixed while moving p_1 in the direction of increasing separation; otherwise we now move both p_1 and p_2 simultaneously so that the line through $\mathbf{p}$ remain continuously tangential to e_i. In both cases, we eventually reach a critical placement belonging to $\mathbf{e}$. **Q.E.D.**

In the following analysis of elementary co-motion, we shall consider three possibilities for a pair $\mathbf{e} = (e_1, e_2)$ of displaced elements, depending on whether the elements are walls or corners. See Figure 5. We shall arrange the coordinate system to place them in "standard position" as follows:

(Wall-wall) So $\mathbf{e}$ is a pair of displaced walls. Let e_1 lie on the x-axis and e_2 lie in the upper half-plane on a line L which passes through the origin. By dividing e_1 into two parts if necessary, we may assume w.l.o.g. that e_1 lie on the positive x-axis.

(Wall-corner) Let e_1 be a displaced wall lying on the x-axis and e_2 a displaced corner which is part of a circle D whose center lies on the positive y-axis.

(Corner-corner)

Let e_i be a part of the circle D_i where D_1 (resp. D_2) is centered at the origin (resp. a point on the positive y-axis).

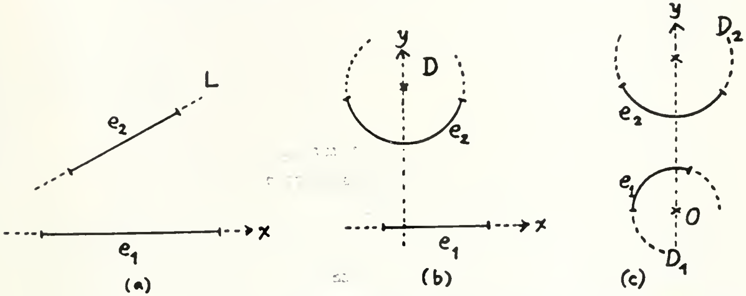


Figure 5. Standard positions for a pair of displaced elements.

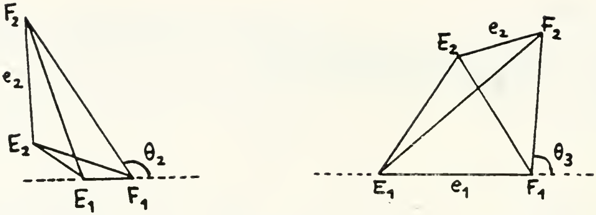
For any placement (or pair of points) $\mathbf{p} = (p_1, p_2)$ let $\theta'(\mathbf{p})$ denote the angle (measured counter-clockwise from the positive x-axis) that the directed line from p_1 to p_2 makes with the x-axis. Define $\theta(\mathbf{p})$ to be $\min\{ \theta'(\mathbf{p}), \theta'(\mathbf{p}) - \pi \}$. With this definition, we have $0 \le \theta(\mathbf{p}) \le \pi$ and $\theta((p_1, p_2)) = \theta((p_2, p_1))$. The proof of the next lemma is long; the topology involved is elementary but interesting.

Lemma 4. Given stable placements $\mathbf{p}$ and $\mathbf{q}$ belonging to $\mathbf{e}$, it can be decided in constant time whether they are adjacent.

Proof. For the purposes of this proof, the terms 'critical placements', 'co-motions' and 'adjacent' are modified so that they satisfy the original definitions except that any (even if only implicit) assumption of 'freeness' in these definitions are dropped; we refer to original terms by explicitly qualifying them as 'free'. Thus: we now use the terms free critical placements, free co-motions, freely adjacent. For example, when we now say two placements $\mathbf{p}, \mathbf{q}$ are adjacent we just mean that they belong to a common $\mathbf{e}$, without assuming there is a free co-motion between them.

We consider the three possibilities for $\mathbf{e}$ using the "standard position" convention above. Let E_i and F_i denote the left and right endpoints of e_i (note: below, 'left' and 'right' will be made unambiguous for the cases involving displaced corners).

Wall-wall case: Consider the four critical placements E_1E_2 (abbreviation for (E_1, E_2)), E_1F_2, F_1E_2 and F_1F_2. See Figure 6.



(a) $\theta_2 = \theta(F_1F_2)$. (b) $\theta_2 = \theta(E_1E_2)$.

Figure 6. Two possibilities for the wall-wall case.

Let $\theta_1 = \theta(E_1F_2)$, $\theta_2 = \min\{\theta(E_1E_2), \theta(F_1F_2)\}$, $\theta_3 = \max\{\theta(E_1E_2), \theta(F_1F_2)\}$ and $\theta_4 = \theta(F_1E_2)$. Observe that $\theta_i < \theta_{i+1}$, $i = 1, \ldots, 3$. We define a retraction σ taking any placement p belonging to e to σ(p) = p′ (also belonging to e) such that θ(p) = θ(p′) and the line through p′ passes through one of the F_i. Observe that the distance between the discs at p′ is greater than that at p (unless p = p′).

We can now deduce the adjacency structure of placements belonging to e. Consider the set K_i of placements p where σ(p) passes through F_i, $i = 1, 2$. Clearly if p is free then it is freely adjacent to $\sigma(p) \in K = K_1 \cup K_2$. Thus we only have to give the adjacency structure of placements in K.

> **Claim.** Let p = (p_1, p_2) and q = (q_1, q_2) be free placements in $\sigma(K_1)$. Then they are freely adjacent iff the 'obvious' co-motion π(p, q) is free, where π(p, q) can be described as the co-motion which keeps $p_1 = q_1 = F_1$ fixed while p_2 moves directly to q_2.

This comes from the fact that any free elementary co-motion from p to q can be converted (using σ and some 'cutting and pasting') into the form claimed. This immediately implies that there are at most two free-adjacency classes in K_1 and if there are exactly two classes then F_1E_2 and F_1F_2 are not freely adjacent. Since the same holds for K_2, it follows that every free placement in K belongs to at most three free-adjacency classes whose respective representatives are F_1E_2, F_1F_2 and E_1F_2.

Wall-corner case: Recall the standard position convention. We will initially modify (e_1, e_2) to simplify the description: Consider the points P and Q on the circle D such that the line E_1P (resp. F_1Q) is tangential to D at P (resp. Q) and D lies to the right (resp. left) of this line. Let α be the lower of the two arcs of D bounded by P and Q. If e_2 does not intersect the arc α then there are no stable placements belonging to e; we may therefore assume otherwise. We shall replace e_2 by $e_2 \cap \alpha$. Hence E_2 is possibly replaced by P and F_2 possibly by Q. We make an analogous modification to e_1: let P' be the point on the x-axis such that P' is the leftmost point on e_1 such that (P', E_2) is a stable placement. Similarly Q' is the rightmost point on e_1 such that (Q', F_2) is a stable placement. (Note that P', Q' exist after the first modification.) Clearly, we may again replace e_1 by $e_1 \cap P'Q'$. We now define six critical placements denoted by $\square_i$ and $\square'_i$ $(i = 1, 2, 3)$. These are not necessarily distinct:

a. $\square_1 = (E_1, E_2)$ and $\square'_1 = (F_1, F_2)$.

b. If (E_1, F_2) is stable then $\square_2 = \square'_2 = (E_1, F_2)$. Otherwise, let $E'_1 \in D$ (resp. $F'_2 \in x-$axis) such that the tangent to e_2 at E'_1 (resp. F_2) passes through E_1 (resp. F'_2) and D lies to the left of the tangent. Note that $E'_1 \in e_2$ (resp. $F'_2 \in e_1$) because of our initial modifications to e. Define $\square_2 = (E_1, E'_1)$ and $\square'_2 = (F'_2, F_2)$.

c. $\square_3$ and $\square'_3$ are defined as in b. except that the roles of the E's and F's are exchanged and 'left' becomes 'right'.

See Figure 7 for some examples.

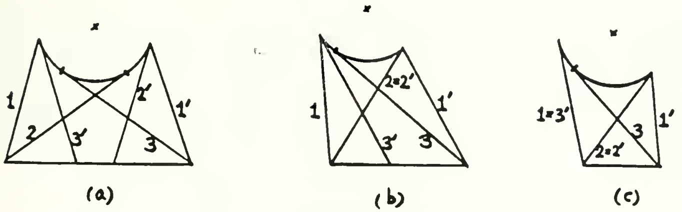
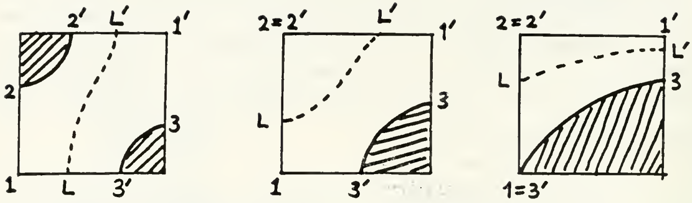


Figure 7. Some wall-corner cases.

We shall parametrize e_i $(i = 1, 2)$ (regarding it as a curve rather than just a set of points) so that $e_i = e_i(t)$, $t \in [0, 1]$, $e_i(0) = E_i$, $e_i(1) = F_i$. The set of placements belonging to (e_1, e_2) is parametrized by the unit square $[0, 1] \times [0, 1]$. The six critical placements defined above lie on the boundary of this unit square: see Figure 8 for the parametrization of the examples in Figure 7.



(a) (b) (c)

Figure 8. Parametrized representations of the wall-corner cases.

The shaded regions in Figure 8 represent unstable placements. The boundary of the (unshaded) stable region is partly formed from the boundary of the unit square and partly from the unique paths from $\square_2$ to $\square'_2$ and from $\square_3$ to $\square'_3$ which move by remaining continuously tangential to e_2. Note that when the 6 placements are distinct, their clockwise cyclic order on the boundary is

$$\square_1\square_2\square'_2\square'_1\square_3\square'_3.$$

Moreover, when they are not distinct, this cyclic order is still valid after elimination (or identification) of duplicate placements.

For any $p_2 \in e_2$, let p_2^L be the leftmost point on e_1 such that (p_2^L, p_2) is stable. p_2^R is similarly defined as the rightmost point. Note that $\theta(p_2^L, p_2) < \theta(p_2^R, p_2)$. We now define a retraction σ which takes any $\mathbf{p} = (p_1, p_2)$ to $\sigma(\mathbf{p}) = (p'_1, p_2)$ where

$$p'_1 = \begin{cases} p_1 & \text{if the line } p_1p_2 \text{ is vertical} \\ p_2^L & \text{if } \theta(p_1, p_2) < \pi/2 \\ p_2^R & \text{if } \theta(p_1, p_2) > \pi/2. \end{cases}$$

Note that (a) the retraction never decreases the distance between the two discs, (b) $\theta(\mathbf{p}) < \pi/2$ iff $\theta(\sigma(\mathbf{p})) < \pi/2$, and (c) if the line $\mathbf{p}$ is not vertical then $\sigma(\mathbf{p})$ is on the boundary of the stable region. The set of placements $\mathbf{p}$ such that the line through $\mathbf{p}$ is vertical forms a curve in *FP* (this is especially easy to see in the parametrized subspace). We may call it the 'watershed' for obvious reasons: this curve is indicated by a dotted line in Figure 8. The watershed (when non-empty) divides the boundary of the stable region into a 'right-hand' and a 'left-hand' boundary. With respect to the parametrization by the unit square, the retraction takes any placement to the right (left) of the watershed *horizontally* across to the right-hand (left-hand) boundary of the stable region. Furthermore, if a placement p in the watershed is free then it is freely adjacent to the two placements on the boundary of the stable region which lie on the 'horizontal level' of p (in the parametrized space). Clearly there are at most two free-adjacency classes in the watershed (since distance between the discs along the watershed is unimodal with a unique minimum). As we shall see, these facts reduce the free-adjacency structure of the stable region to that of its boundary.

If the watershed is non-empty, then the two extreme placements bounding the watershed will be denoted $\square_L$ (the leftmost) and $\square_R$ (the rightmost). Observe that $\square_L$ lies on the segment of the boundary of the stable region represented by $\square_2\square_1\square'_3$. Similarly $\square_R$ lies in the segment $\square'_2\square'_1\square_3$.

Now consider the set of placements p belonging to e such that the line through p passes through the center of *D*. If this set is non-empty, then it again forms a curve in *FP*. Let the placements at the extremes of this curve be denoted $\square'_L$ and $\square'_R$. Again, $\square'_L$ (resp. $\square'_R$) lies on the boundary of the stable region represented by $\square_2\square_1\square'_3$ (resp. $\square'_2\square'_1\square_3$). We note three possibilities for the relative positions of $\square_L$ and $\square'_L$ on the boundary of the stable region (see Figure 9).

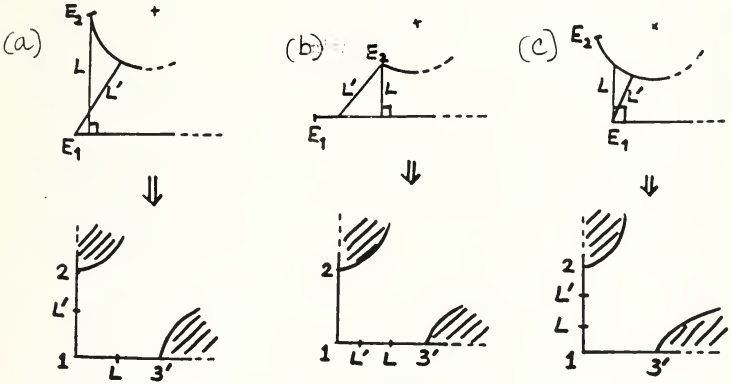


Figure 9. Relative positions of $\square_L$ and $\square'_L$.

a. $\square_L$ lies on the segment $\square_1\square'_3$ and $\square'_L$ lies on the segment $\square_1\square_2$.

b. Both $\square_L$ and $\square'_L$ lie on the segment $\square_1\square'_3$. Then note that $\square'_L$ lies between $\square_1$ and $\square_L$.

c. Both $\square_L$ and $\square'_L$ lie on the segment $\square_1\square_2$. Then note that $\square_L$ lies between $\square_1$ and $\square'_L$.

First we consider the free-adjacency structure of free placements in $\square_L\square'_3$ (resp. $\square'_L\square_2$). Suppose case a. holds: Then any free placement on the segment $\square_L\square_1\square'_L$ is freely adjacent to $\square_1$. Any free placement on the segment $\square_L\square'_3$ (resp. $\square'_L\square_2$) is freely adjacent to $\square'_3$ (resp. $\square_2$). Thus there are at most three free-adjacency classes in this case. Next consider case b.: Any free placement on the segment $\square_2\square_1\square_L$ (resp. $\square_L\square'_3$) is freely adjacent to $\square_2$ (resp. $\square'_3$). Thus there are at most two free-adjacency classes. Finally, case c. is similar to case b.

The above assumes that $\square_L$ and $\square'_L$ are defined. However even if one or both are undefined, exactly the same analysis can still be carried out.

We now consider the adjacency structure on the segment $\square_2\square'_2$ (Figure 7(a)). An elementary geometric fact will be useful here: let H be the point on D such that OH (O is the origin) is a tangent of D through O and $\theta(OH) < \pi/2$. If AX is any tangent to D such that X (resp. A) is on the x-axis (resp. D) then the length $|AX|$ of the segment AX increases monotonically with the distance of X from the origin since $|AX|^2 = |OH|^2 + |OA|^2$. Define $\square_H$ to be the placement such that the minimal distance between the discs for placements on this segment ($\square_2\square'_2$) is attained at $\square_H$. If the pair (O, H) belongs to e then the above geometric fact implies that $\square_H = (O, H)$; otherwise $\square_H$ is either $\square_2$ or $\square'_2$. It is now easy to see that every free placement on the segment $\square_2\square'_2$ belongs to at most two free-adjacency classes which can be easily determined. (The idea is to define a retraction of placements in $\square_2\square'_2$ to move away from $\square_H$.)

We finally put all these facts together to deduce the 'global' free-adjacency structure of the boundary of the stable placements union the watershed. The above analysis shows that every free placement on the boundary is freely adjacent to one of the 6 critical placements. Call a horizontal line in the parametrized space a 'free horizontal path' if it represents a free co-motion. We proceed as follows: the set of free placements in the boundary of the stable region consists of at most 6 pairwise disjoint segments. The set of free placements in the watershed consists of at most 2 disjoint segments. These segments (whether in the boundary of the stable region or in the watershed) are easy to determine, using the foregoing discussions. Given a boundary segment α and a watershed segment β, define the binary relationship R such that $\alpha\, R\, \beta$ holds iff they are connected by some free horizontal path. It is easy to check this if $\alpha\, R\, \beta$ holds for any α and β. The reflexive, transitive closure $\bar{R}$ of the relation R gives us the global free-adjacency structure of free placements on the boundary of the stable region or the watershed. More precisely,

Claim. Two free placements **p** and **q** in the watershed or the boundary of the stable region are freely adjacent iff their respective segments α and β satisfy $\alpha \overline{R} \beta$.

To see this, take any free elementary co-motion π from **p** to **q**. Then the retracted co-motion $\sigma(\pi)$ is seen to determine a sequence of segments where consecutive segments are connected in the above sense. This proves the claim and the wall-corner case.

Corner-corner case: It turns out that this case introduces no additional complications. The reader can easily go through all our constructions and analysis of the wall-corner case almost verbatim.

This concludes the proof of lemma 4. Q.E.D.

For each critical placement **p**, there are $O(1)$ other critical placements that it could be adjacent to. A size-bounded set $S(\mathbf{p})$ of potentially-adjacent placements for any given critical placement **p** can be obtained from the graphs G_i in constant time. The preceding lemma then implies

Corollary. The adjacency relationship between critical pairs can be computed in time $O(n^2)$.

5. Analysis of discontinuities

We will investigate the discontinuities in stable co-motions. The first task is to show that we may restrict ourselves to stable co-motions with a finite number of discontinuities. We make a general remark. The reader may avoid some of the lemmas of this section (in particular, proceed directly to lemma 7) if he is willing to make one of two assumptions: either (a) that co-motions are reasonably smooth (say, piecewise algebraic) or (b) appeal to a general fact in differential topology (Sard's lemma [SS1]) which essentially says that we can always perturb a path to avoid local singularities. Nevertheless, it is instructive to give direct proofs of these results.

Definition. Let π be any co-motion, **p** a placement and $t_0 \in [0,1]$. We say π *approaches* **p** *at* t_0 if there exists a sequence t_n $(n \geq 1)$ such that $t_n \to t_0$ and $\pi(t_n) \to \mathbf{p}$.

Surely π approaches $\pi(t)$ at t for all t (take the sequence $t_n = t$ for all $n \geq 1$). But it is not hard to see that π may approach more than one placement at some t.

Lemma 5. Let π be any co-motion. If π approaches **p** at t_0 then the line through **p** coincides with the line through $\pi(t_0)$. If π is also stable, then there exists a continuous co-motion from **p** to $\pi(t_0)$.

Proof: The first part is clear. For the second part, let $\mathbf{p} = (p_1, p_2)$ and $\pi(t_0) = (q_1, q_2)$. Consider any motion π'_i ($i = 1, 2$) of B_i which moves p_i to q_i in a direct line: note that π'_i is indeed a motion because the line segment joining p_i and q_i is free. To complete the proof, it is enough to show that π' defined as the pair (π_1, π_2) is a co-motion, i.e. for all t, $\pi'(t)$ is free. To see this, since π is stable, let $\pi = Im(\pi'')$ for some continuous π''. Then $\pi''(t_0)$ is free. It is easy to see that for all t, the line segment defined by $\pi'(t)$ contains the segment defined by $\pi''(t_0)$, so $\pi'(t)$ is free. **Q.E.D.**

Lemma 6. Let π be any stable co-motion and $\mathbf{p}$ a stable placement. Then the set

$$J_{\pi}(\mathrm{p}) = \{t \in [0, 1] : \exists \mathrm{q} \text{ adjacent to } \mathrm{p} \text{ such that } \pi \text{ approaches } \mathrm{q} \text{ at } t\}.$$

is closed.

Proof. Let $t_n \to t_0$ ($n \ge 1$), $t_n \in J_{\pi}(\mathrm{p})$. We have to show that $t_0 \in J_{\pi}(\mathrm{p})$. Let π approach q_n at t_n and q_n is adjacent to $\mathbf{p}$. By going to a subsequence, we may assume that q_n belongs to some pair (e_1, e_2) of displaced elements. This implies that $\mathrm{q}_n \to \mathrm{q}_0$ for some q_0 belonging to (e_1, e_2). By going to a subsequence we may assume that $|q_0 - q_n| < 1/n$ for all n. Furthermore, it is clear that q_0 is adjacent to $\mathbf{p}$. It remains to show that π approaches q_0 at t_0: for each n, let $t_{n,i} \to t_n$, ($i \ge 1$) such that $\pi(t_{n,i}) \to \mathrm{q}_n$. We may pick $i(n)$ such that $|t_{n,i(n)} - t_n| < 1/n$ and $|\pi(t_{n,i(n)}) - \mathrm{q}_0| < 1/n$. Now observe that $t_{n,i(n)} \to t_0$ and $\pi(t_{n,i(n)}) \to \mathrm{q}_0$, ie., π approaches q_0 at t_0. **Q.E.D.**

Lemma 7. If there is a continuous co-motion π from p' to q' then there is one between the same pair of placements whose image has a finite number of discontinuities.

Proof. Let $\mathbf{p}$ be any critical placement. Below, we transform π to another continuous π' such that there is some closed interval $I \subseteq [0, 1]$ such that (a) π' is identical to π outside of I, (b) π' has a finite number of discontinuities in I and (c) for any t, if $\pi'(t)$ is adjacent to $\mathbf{p}$ then $t \in I$. By repeating this transformation $O(n^2)$ times (for each critical $\mathbf{p}$) we obtain a continuous co-motion satisfying the lemma.

We now describe the transformation (relative to some critical $\mathbf{p}$) from π to π'. Let $Im\pi$ be the image of π. Let $t_0 = \min J_{Im\pi}(\mathrm{p})$, $t_4 = \max J_{Im\pi}(\mathrm{p})$. If $t_0 = t_4$, we let $\pi' = \pi$. Otherwise, arbitrarily pick t_i ($i = 1, 2, 3$) such that $t_i < t_{i+1}$ ($i = 0, \ldots, 3$). Let $Im\pi$ approach q_j ($j = 0, 4$) at t_j. By the previous lemma, we may pick q_j so that it is adjacent to $\mathbf{p}$. We let π' be identical to π outside $I = [t_0, t_4]$. In the interval I, let π' move continuously from $\pi(t_0)$ (at t_0) to q_0 (at t_1); and from thence to $\mathbf{p}$ (at t_2); and to q_4 (at t_3); and finally to $\pi(t_4)$ (at t_4). Observe that the image of π' has $O(n)$ discontinuities in the interval $[t_0, t_1]$ and in $[t_3, t_4]$; otherwise it has no discontinuities within the interval

I. (Remark: the $O(n)$ discontinuities comes from the fact that a line may be tangential to $O(n)$ displaced corners.) Q.E.D.

We may restrict attention from now on to continuous co-motions whose image has a finite number of discontinuities. Our remaining task in this section is to reduce the discontinuities to a special form.

Lemma 3. If $t_0 \in [0,1]$ is an isolated discontinuity of a stable π then $\pi(t_0^-)$ (unless $t_0 = 0$) and $\pi(t_1^+)$ (unless $t_0 = 1$) are well-defined.

Note that $\pi(t_0)$, $\pi(t_0^-)$ and $\pi(t_0^+)$ may be all distinct: such an example is furnished by the case where $\pi(t_0)$ is co-tangent to two displaced corners.

Proof. Let L be the line through $\pi(t_0)$ and let X_i ($i = 1, 2$) be the set of points x in L such that x lies on some displaced element e of $FP(B_i)$; furthermore, if x is in the interior of e then e is not contained in L. The last clause ensures that if e is contained in L, then only the end-points of e are in X_i. So clearly X_i is a finite set. For each $x \in X_i$, pick a neighborhood $N(x)$ of x such that for distinct x and y in X_i, the Hausdorff distance $d(N(x), N(y)) > 0$. Choose an interval $I = [t_0 - \delta, t_0 + \delta]$ such that for all t in I, $\pi(t)$ belongs to $(N(x_1), N(x_2))$ for some $x_i \in X_i$. Since t_0 is an isolated discontinuity, we may assume that I contains no other discontinuity besides t_0. We now show that $\pi(t_0^-)$ exists, the case of t_0^+ being similar. First note that for δ small enough, there exists a (x_1, x_2) ($x_i \in X_i$) such that for all t in $[t_0 - \delta, t_0)$, $\pi(t)$ belongs to $(N(x_1), N(x_2))$; otherwise there would be more than one discontinuity in I. From this it follows from the geometry of the situation that for all $\epsilon > 0$ small enough, there is a δ', where $\delta > \delta' > 0$, such that if $t \in [t - \delta', t_0)$ then $|\pi(t) - (x_1, x_2)| < \epsilon$. Thus $\pi(t_0^-)$ is defined and equals (x_1, x_2). Q.E.D.

We shall be interested in isolated discontinuities of a very special form, given next. The discontinuities come in two flavors: one in which a disc 'jumps' away from a corner and the other where a disc 'slides' along a displaced wall.

Definition. Let $t \in (0,1)$ be an isolated discontinuity of a stable π. Let $\pi(t^-) = (p_1, p_2)$ and $\pi(t^+) = (q_1, q_2)$. Then t is a B_1*-jump* if

(a) $\pi(t)$ is critical,

(b) $\pi(t) \in \{\pi(t^-), \pi(t^+)\}$,

(c) $p_2 = q_2$ and

(d) the open line-segment (p_1, q_1) lies in the interior of $FP(B_1)$.

The discontinuity t is a B_1*−slide* if it satisfies the following

(c) the closed line-segment $[p_1, q_1]$ lies in some displaced wall e of B_1.

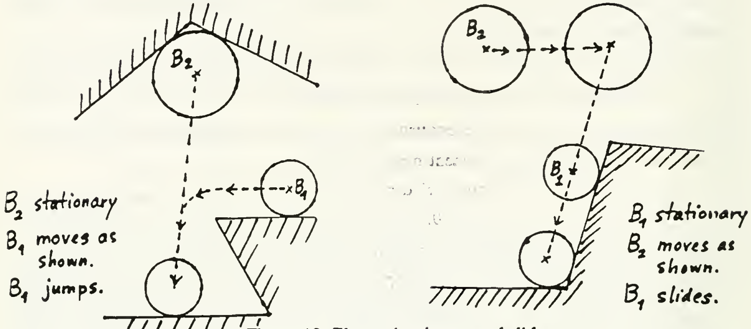


Figure 10. Illustrating jumps and slides.

A B_2*−jump* is defined analogously. A *jump* (discontinuity) of π is either a B_1- or a B_2-jump. A *jump−type* is a pair of placements of the form $(\pi(t^-), \pi(t^+))$ for some jump t of some stable π. Similarly, a *slide* is either a B_1- or a B_2-slide. Observe that as a consequence of our definition of Im, if B_1 is sliding along a displaced wall e then the image of B_1 is necessarily an end-point of e (even though we could have imagine B_1 'sliding' right pass the end).

Lemma 9. There are $O(n^2)$ jump-types.

Proof. There are $O(n^2)$ critical placements and each critical placement can give rise to at most two jump-types. **Q.E.D.**

We next show that slides can always be removed. For the next two results, we adopt the following notations: π' is a continuous co-motion from $\mathbf{p}'$ to $\mathbf{q}'$; π is the image of π' and is thus a stable co-motion from $\mathbf{p} = Im(\mathbf{p}')$ to $\mathbf{q} = Im(\mathbf{q}')$; write $\pi' = (\pi'_1, \pi'_2)$ and $\pi = (\pi_1, \pi_2)$.

Lemma 10. Let π' be a continuous co-motion from $\mathbf{p}'$ to $\mathbf{q}'$. Then there exists a continuous co-motion π'' between the same pair of placements whose image $Im(\pi'')$ has no slides.

Proof. Suppose t_0 is a B_1-slide of $\pi = Im(\pi')$. We show how to modify π' to remove the slide. By definition of Im, for all t, $\pi_i(t)$ is on the boundary of $FP(B_i)$, $i = 1, 2$. We omit the (routine) proof of the following

Claim: $\pi_1(t_0) = \pi'_1(t_0)$; in particular, $\pi'_1(t_0)$ is on the boundary of $FP(B_1)$.

From this claim we can modify π' such that its modified image has no slide discontinuities (except possibly at $t = 0$ or 1) simply by ensuring that $\pi'_i(t)$ is not on the boundary of $FP(B_i)$ for $i = 1, 2$; this is done simply by a perturbation of π'.

It remains to show how to remove slide discontuities of the form $t_0 = 0$ or 1. Suppose $t_0 = 0$ is a B_1-slide. Clearly the line through $\pi'(0)$ is tangent to some displaced wall e at $\pi'_1(0)$. We consider two possibilities. Case (a): $\pi_1(0) \neq \pi'_1(0)$. (Note that in this case $\pi'_1(0)$ is not a vertex.) We perturb a small initial portion of π' by first translating $\pi'(0)$ to some neighboring placement q which directly away from the wall e and parallel to e. Then the original co-motion π' can continue from q with minimal modifications, and its image π has no slide at $t = 0$. Case (b): $\pi_1(0) = \pi'_1(0)$. (Note that $\pi'_1(0)$ must be a vertex where a displaced wall meets a displaced corner. This is really a technical consequence of our definition of the map *Im*.) We first move $\pi'_2(t)$ for t in a small neighborhood $[0,\delta]$ of 0 (while keeping $\pi'_1(t)$ fixed at $\pi'_1(0)$) so that the line through $q = (\pi'_1(0), \pi'_2(\delta))$ is not parallel to e. Then the original π' can continue from q with minimal modifications. **Q.E.D.**

The main result for this section can now be proved.

Theorem 11. Let p′ and q′ be free placements. There is a continuous co-motion from p′ to q′ iff there exists a stable co-motion from $\mathbf{p} = Im(\mathbf{p}')$ to $\mathbf{q} = Im(\mathbf{q}')$ with $O(n^2)$ jumps (but no other discontinuities).

Proof. Any stable co-motion from $Im(\mathbf{p}')$ to $Im(\mathbf{q}')$ with only isolated discontinuities can clearly be converted into a continuous co-motion from p to q. Conversely, let π' be a continuous co-motion from p′ to q′ and $Im(\pi') = \pi$. By the above lemmas, we may assume that all the discontinuities of π are isolated and there are no slide discontinuities. We will modify π' below that each discontinuity of $Im(\pi')$ is replaced by at most n jumps. Since we can eliminate all jumps but one among each jump-type, the theorem follows.

Let t_0 be an isolated discontinuity, but not a jump, of π. First let us assume that t_0 is not 0 or 1. We now obtain another continous co-motion $\pi'' = (\pi''_1, \pi''_2)$ by modifying π' about some interval $I = [t_0 - \delta, t_0 + \delta]$. We making δ small enough, we may assume that t_0 is the only discontinuity inside I, $\pi'(t_0^-)$ is adjacent to $\pi'(t_0 - \delta)$ and $\pi'(t_0^+)$ is adjacent to $\pi'(t_0 + \delta)$. Let $\pi(t_0^-)$ (resp. $\pi(t_0^+)$) belong to a pair of displaced elements $\mathbf{e} = (e_1, e_2)$ (resp. $\mathbf{e}' = (e'_1, e'_2)$. To expose the essential ideas of the proof, we initially assume

$$\pi_2(t_0^-) = \pi_2(t_0^+),$$

and so $e_2 = e'_2$. Consider the line segment α between $\pi_1(t_0^-)$ and $\pi_1(t_0^+)$. Clearly α intersects only a finite number of points on displaced corners of $FP(B_1)$. We may assume w.l.o.g. that

$$\pi_1(t_0^-) = p_0, p_1, \ldots, p_k = \pi_1(t_0^+), \quad (k \geq 0)$$

are these points of intersection in order of increasing distance from $\pi_2(t_0^-)$. Of course, α may contain an entire displaced wall but then only the endpoints of the displaced wall appears in the above sequence. See Figure 11.

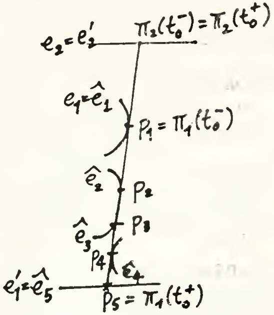


Figure 11. Intersection of α with the boundary of $FP(B_1)$.

We will first show how to reduce the case where $k > 1$ to the case $k = 1$. Say α is tangential to the displaced corner e_l at p_l for $l = 1, \ldots, k-1$. It is not hard to see that there are essentially two possibilities for arranging the e_l with respect to α: either all the e_l's lie on one side of α or for some index $l_0 = 0, \ldots, k-1$, we have that $e_0, \ldots, e_{l_0}$ lie on one side of α while the rest lie on the other side. To describe the modified π'' around the interval I, we first move B_2 from $\pi'_2(t_0 - \delta)$ to $\pi_2(t_0 - \delta)$ while π''_1 moves from $\pi'_1(t_0 - \delta)$ to $\pi_1(t_0 - \delta)$. (Note that since $\pi(t_0^-)$ belongs to $\mathbf{e}' = (e'_1, e'_2)$, $\pi_1(t_0 - \delta) \in e'_1$.) Then π_2 and π_1 moves as in π, viz., from $\pi(t_0 - \delta)$ to $\pi(t_0^-)$. Then π''_2 remain fixed while π''_1 moves along the segment α from $\pi_1(t_0^-)$ to $\pi_1(t_0^+)$. We basically reverse this process so that at time $t_0 + \delta$, π'' again agrees with π'. This is illustrated in the Figure 12. This completes the present modification. Consider the image of the modified co-motion. Clearly it has k discontinuities where the disc B_1 'leaps' from p_{l+1} to p_l ($l = 0, \ldots, k-1$). Furthermore all but the p_k to p_{k-1} leap are B_1-jumps since the pairs (p_l, p_{l+1}) are critical for $l = 0, \ldots, k-1$.

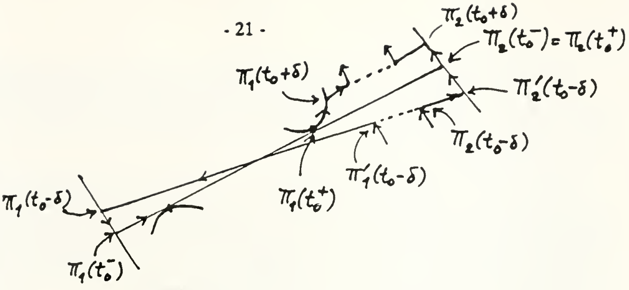


Figure 12. Reducing the case $k > 1$ to $k = 1$.

After the preceding modifications, we may restrict ourselves to the case $k = 1$ (Note: $k = 0$ is easy - this is where $\pi_1(t_0^-) = \pi_1(t_0^+) \neq \pi_1(t_0)$). Let $\pi_2(t_0^-) = p_{-1}$. Note that p_0 lies in the line segment $\beta = [p_{-1}, p_1]$. Consider an elementary co-motion which moves the (endpoints of the) segment β such that β remains tangential to e_1 until the first time β reaches the position $\beta' = [p'_{-1}, p'_1]$ with the following property: first note that since this elementary co-motion maintains the tangential property with respect to e_1, it also traces a motion of the point p_0 to the point p'_0 at which the line $p'_{-1}p'_1$ is tangential to e_1. The required property of β' is that either (p'_1, p'_{-1}) or (p'_0, p'_{-1}) is a critical placement. See Figure 13.

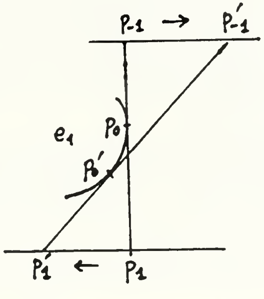


Figure 13. Modifying the discontinuity when $k = 1$.

The desired co-motion π'' inside the interval I (for the case $k = 1$) can now be described: in the interval $[t_0 - \delta, t_0)$, we move the discs along the path prescribed by the above elementary co-motion for p_{-1} and p_1 until they reach the placement (p'_{-1}, p'_1). At that point, the disc B_2 remain fixed at p_{-1} while B_1 moves in a straightline to p'_0; the two discs now 'retrace' their paths from (p'_0, p'_{-1}) back to (p_0, p_{-1}).

The case where $\pi'_2(t_0^-)$ and $\pi'_2(t_0^+)$ are not equal can be handled along the same lines as above. Finally, we have to consider the case where $t_0 = 0$ or 1. These details can be left to the reader. **Q.E.D.**

A placement q is called a *complement* of a critical placement p if the pair (p, q) is a jump type. Let the *canonical coordinated motion graph CCG* be the (undirected) graph whose nodes are the critical placements or complements of such, and whose edges join two placements iff the placements are adjacent or if one is a complement of the other. From the foregoing results, *CCG* has $O(n^2)$ edges and vertices.

> **Corollary.** Let p and q be free placements which are adjacent to p′ and q′, respectively. There is a continuous co-motion from p to q iff there is a path in the canonical coordinated motion graph *CCG* from p′ to q′.

6. The Algorithm for Two Discs

We describe the algorithm in two phases.

Preprocessing.

The input is a description of S and r_1, r_2.

1. Compute the graphs G_1, G_2 in time $O(n \log n)$.
2. For each pair of displaced elements $\mathbf{e} = (e_1, e_2)$ where e_i is from G_i, we compute the size-bounded set $C(\mathbf{e})$ of critical placements and complements of critical placements belonging to it. For each placement p in $C(\mathbf{e})$, compute the size-bounded subset $N(\mathbf{p}, \mathbf{e}) \subseteq C(\mathbf{e})$ of placements which are either adjacent to p or is the complement of p. This step takes a total of $O(n^2)$ time.
3. Construct the canonical coordinated motion graph *CCG* from the results in step 2. In particular, the vertex set V is the union of the sets $C(\mathbf{e})$ over all **e**. The edge set E is easily constructed from the sets $N(\mathbf{p}, \mathbf{e})$. This takes time $O(n^2)$.

Searching.

The input is the graph *CCG* and the initial and final placements, p, q.

1. Verify that p and q are free, otherwise there is no path. If free, compute the images $Im(\mathbf{p})$ and $Im(\mathbf{q})$, and the pair of displaced elements **e** that they belong to. Note that $Im(\mathbf{p})$ can be computed in linear time in a straighforward manner.
2. Compute any pair of critical placements p′ and q′ that are adjacent to $Im(\mathbf{p})$ and $Im(\mathbf{q})$, respectively. Determine the vertices u and v in V which represent the critical

placements **p**′ and **q**′. This takes constant time.

3. Search for a path in CCG connecting u and v, in time $O(n^2)$.

7. Coordination of Three Discs

Consider any free placement $\mathbf{p} = (p_1, p_2, p_3)$ of the three discs. First, assume that triangle $\Delta\mathbf{p}$ is non-degenerate. We define the *Im* map by imagining a mutually repulsive force concentrated at the in-center c of the triangle $\Delta\mathbf{p}$ which moves each of the discs directly away from c until the first time each disc touches the obstacle S. If the triangle $\Delta\mathbf{p}$ is degenerate, *Im* is undefined. Proceeding as before, we can define the notion of elementary co-motion and show that there are $O(n^3)$ elementary co-motions. The image of a continuous co-motion has discontinuities which can be reduced to two canonical forms: jumps (which are generalizations of the 2 discs case) and flips (this is new and arises from the three discs getting aligned). Again, we show that there is a continuous co-motion between two placements iff there is a co-motion that is composed of a sequence of elementary co-motions possibly connected by jumps and flips. Thus motion planning is reduced to searching a suitable graph. In the interest of keeping this paper compact, we shall only sketch the solution.

We define a placement (p_1, p_2, p_3) to be *stable* if each p_i lies on the boundary of $FP(B_i)$. We say a co-motion π is *stable* if for all $t \in [0, 1]$, $\pi(t)$ is stable. Note that this definition is simplier than the corresponding one for two discs which insists that π must be the image of some continuous co-motion; the simplicity simplifies some of the constructions. Also our definition of the image map is such that if **p** is stable then $Im(\mathbf{p}) = \mathbf{p}$. We say a co-motion $\pi = (\pi_1, \pi_2, \pi_3)$ is *nice* if (a) π is the image of a continuous co-motion, (b) π has a finite number of discontinuities, and (c) for each $t \in [0, 1]$, both $\pi(t^-)$ (unless $t = 0$) and $\pi(t^+)$ (unless $t = 1$) are defined and $\pi(t) = \pi(t^-)$ or $\pi(t^+)$. We can show that any co-motion can be converted to a nice one, but nothing essential is lost even if we begin with only nice co-motions. Define a placement $\mathbf{p} = (p_1, p_2, p_3)$ to be *critical* if at least 3 of the following constraints hold, $(i, j = 1, 2, 3)$:

$(a)_{i,j} d(p_i, p_j) = r_i + r_j$.

$(b)_i$ p_i is a vertex of G_i.

$(c)_{i,j}$ the line $p_i p_j$ is tangent to a displaced corner of G_i at p_i.

A continuous co-motion is *elementary* if for some triple **e** of displaced elements, the co-motion lies entirely within **e**. Two placements **p** and **q** are *adjacent* if there is an elementary co-motion between them.

Lemma 12. *There are $O(n^3)$ critical placements and for every placement* p *we can find in constant time a critical placement which is adjacent to* p.

The fact of $O(n^3)$ critical placements comes from the fact that for any triple e of displacements elements, there are $O(1)$ critical placements belonging to e. To prove the second part of the lemma we show that for any placement p, we can move from p to another q using some elementary co-motion so that the number of conditions (namely, $(a)_{i,j}$, $(b)_i$ and $(c)_{i,j}$ in the above definition of critical placements) satisfied by p is increased.

Lemma 13. *For each pair of critical placements, we can decide in constant time whether they are adjacent.*

The proof of this lemma involves tedious though elementary analysis of the topology of placements belonging to triples of displacements elements.

The next task is to simplify discontinuities. Let π be a nice co-motion such that t is a discontinuity. We call t a B_i-*jump* if (a) the motion of the other two discs B_j $j \neq i$ are continuous at t (so B_i is the only disc that moves discontinuously at t) and (b) $\pi(t^-)$ or $\pi(t^+)$ are critical. We call t a B_i-*flip* if (a) the motion of the other two discs not equal to B_i are continuous at t, (b) $\pi(t)$ is degenerate with the center of B_i lying in the line segment formed by the other two discs, (c) the placement of the other two discs is critical (as defined for placements of-pairs of discs) and (d) B_i is touching the boundary of $FP(B_i)$. The two crucial lemmas are:

Lemma 14. There are $O(n^3)$ jumps and flips.

Lemma 15. Every nice co-motion can be transformed into one whose only discontinuities are jumps or flips.

We can now construct the canonical co-motion graph *CCG* whose vertices are critical placements and complements of such, and whose edges are elementary co-motions, jumps or flips. This can be done in $O(n^3)$ time. Finally, planning a co-motion between two placements is reduced to a search of this graph.

8. Conclusions

We have shown that the retraction approach again led to more efficient algorithms than was previously obtained using the decomposition approach of [SS1-3]. Our algorithms further demonstrate that retracts *need not* be generalizations of Voronoi diagrams, thereby enlarging the scope for applications of the retraction idea.

It is possible to extend the ideas in this paper for coordinating more discs although the complexity of the analysis rapidly increases: in particular, for four discs we do not see a simple way to define retractions so that the discontinuities can be easily dealt with. One problem is that with four disc, the center of one of the discs may be in the interior of the triangle formed by the centers of the other three. So a natural open problem is to find a general and easy-to-describe method for coordinating an arbitrary but fixed number of discs. In fact, we do not know if the complexity of moving k discs is $O(n^k)$ for $k \geq 4$. Another interesting problem is to generalize the two discs algorithm presented here so that the graph CCG is independent of the radii of the discs: note that the one disc algorithm [OY] has this property.

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